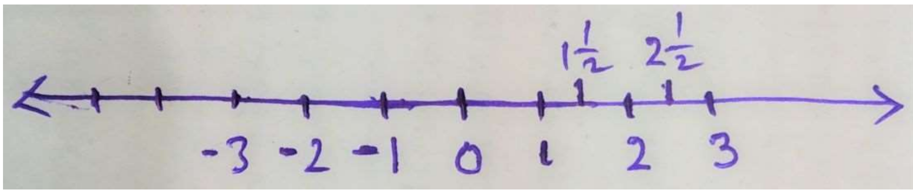
Physics (H) SEM_IV CC VIII (MATHEMATICAL PHYSICS-III: Complex Analysis):

Topics to cover: Brief Revision of Complex Numbers and their Graphical Representation. Euler's formula, De Moivre's theorem

What is complex number (from a mathematician's perspective)?

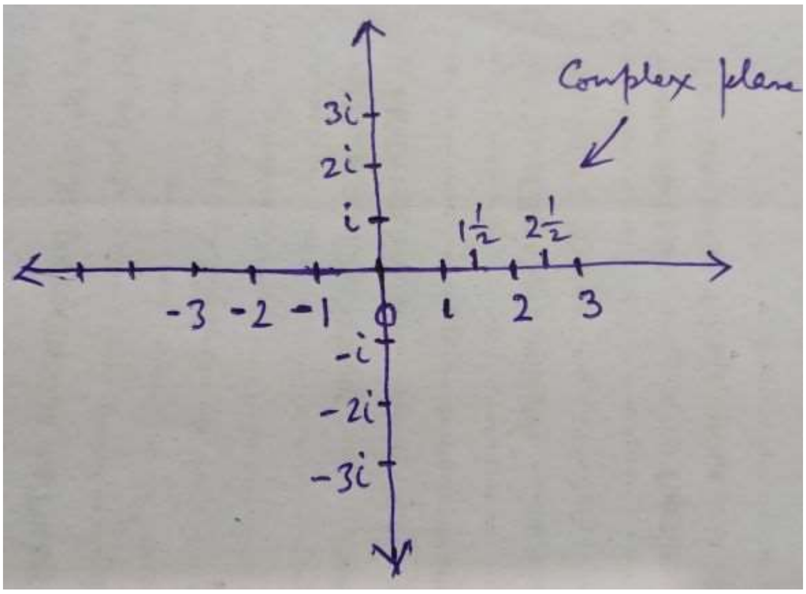
We have seen that the equation $(x^2 + 1) = 0$ has no real solution as $(x^2 + 1) = 0$ gives $x^2 = -1$ and square of every real number is non-negative. So, we need to extend the real number system to a larger system so that we can find the solution of the equations of the form $x^2 = -1$. In fact, the main objective is to solve the equation, $ax^2 + bx + c = 0$, where $D = b^2 - 4ac < 0$, which is not possible in the system of real numbers. Let us denote $\sqrt{-1}$ by the symbol 'i'. Then, we have $(i)^2 = -1$ and any number of the form $(x + iy)$, (where x and y are real numbers and $i = \sqrt{-1}$), is called a complex number in which 'x' is real part and 'iy' is the imaginary part.

Real numbers are represented as points on a one dimensional number line.

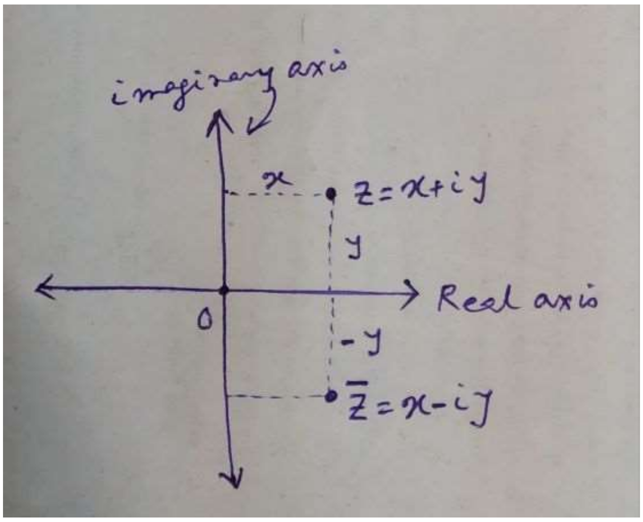


But complex number makes a two dimensional complex plane in which real part is represented along horizontal line and imaginary part is represented along vertical line.

[Note: Argand Diagram: a diagram on which complex numbers are represented geometrically using Cartesian axes, the horizontal coordinate representing the real part of the number and the vertical coordinate the complex part. The complex plane is also called Argand plane.]



A complex number $z = x + iy$ is represented as a point on this plane. Another point represented by $\bar{z} = x - iy$ is called the complex conjugate of the complex number z.



Hence real part of the complex number *z*, *Re(z)*$= x = \frac{z+\bar{z}}{2}$ and imaginary part of the complex number, $Im(z) = y = \frac{z-\bar{z}}{2}$.

Furthermore, one can defines the '**absolute value**' of a complex number *z* as

$$|z| = \sqrt{x^2 + y^2} = \sqrt{(x+iy)(x-iy)} = \sqrt{z\bar{z}}$$

Also to note $|z| = |\bar{z}|$

Note: A real number is a special case of complex number. A real number *x* can be regarded as a complex number *x+i0*, whose imaginary part is 0. A purely imaginary number *iy* is a complex number 0 +*iy,* whose real part is zero.

Complex algebra (fundamental operations with complex numbers:

(1) *Addition*

$$(a+bi)+(c+di) = a+bi+c+di = (a+c)+(b+d)i$$

(2) *Subtraction*

$$(a+bi)-(c+di) = a+bi-c-di = (a-c)+(b-d)i$$

(3) *Multiplication*

$$(a+bi)(c+di) = ac+adi+bci+bdi^2 = (ac-bd)+(ad+bc)i$$

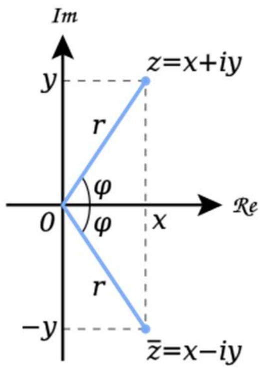
(4) *Division*

If $c \neq 0$ and $d \neq 0$, then

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} = \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2}$$
$$= \frac{ac+bd+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Polar representation of imaginary numbers:

As we see from the figure that a complex number z in complex plane (Argand plane) can also be represented by a vector (blue arrow). This vector can be expressed either by a cartesian ordered pair (x, y) or by a polar ordered pair (r, φ)



Here, $r = |z| = \sqrt{x^2 + y^2}$ and $\varphi = tan^{-1}(\frac{y}{x})$;

We now can express

$$x = |z| \cos \emptyset$$

and

$y = |z| \sin \emptyset$;

Hence

$z = x + iy = |z| \cos \emptyset + i|z| \sin \emptyset = |z|e^{i\emptyset}$ (using Euler's formula)

and

$\bar{z} = x - iy = |z| \cos \emptyset - i|z| \sin \emptyset = |z|e^{-i\emptyset}$ (using Euler's formula)

Now let us do some problems.

Problem set-I: For each complex number given below write the polar form of the number and it's complex conjugate and draw them:

1. $1 + i$
2. $i - 1$
3. $1 - i\sqrt{3}$
4. $-\sqrt{3} + i$
5. $2i$
6. $-4i$
7. -1
8. 3
9. $2i - 2$
10. $2 - 2i$
11. $2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$
12. $4\left(\cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3}\right)$
13. $\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$
14. $2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$
15. $\cos \pi - i \sin \pi$
16. $5(\cos 0 + i \sin 0)$
17. $\sqrt{2}e^{-i\pi/4}$
18. $3e^{i\pi/2}$
19. $5(\cos 20^\circ + i \sin 20^\circ)$
20. $7(\cos 110^\circ - i \sin 110^\circ)$

Note: Polar form of complex number is much more useful in Physics.

Euler's formula: The equation $e^{i\theta} = \cos\theta + i\sin\theta$ is called Euler's formula.

The most famous outcome of Euler's formula is, $e^{i\pi} = -1$;

Also to remember, $e^{i\frac{\pi}{2}} = i$

$e^{in\pi} = (-1)^n$ and $e^{i(2n+1)\frac{\pi}{2}} = (-1)^n i$

De Moivre's theorem:, a theorem which states that *$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$*
It allows complex numbers in polar form to be easily raised to certain powers.

Problem set-II:

1. Prove Euler's theorem using following infinite series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \qquad \text{all } x;$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \qquad \text{all } x;$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \qquad \text{all } x;$$

2. Prove De Moivre's theorem using Euler's formula.

End note: Role of complex number in Physics (complex number from a Physicist's perspective)?

It is an interesting observation that whenever 'i' is multiplied with any number (real or complex) it rotates the number by an angle $\frac{\pi}{2}$ in anti-clockwise direction in the complex plane without changing the magnitude of the number. Similarly whenever '$-i$' is multiplied with any number (real or complex) it rotates the number by an angle $\frac{\pi}{2}$ in clockwise direction in the complex plane without changing the magnitude of the number Not only that multiplication of suitable complex number can produce any short of rotation, i.e., rotation to any value in any direction (clockwise or counter clockwise). It is such a feature that real numbers do not have. That's why Physicists always use complex numbers whenever there is a need to deal simultaneously with magnitude and phase of some physical entity. For example, complex numbers and techniques of complex analysis is used to sove problems related to response of LCR circuits with varying current or voltage, A.C. circuits, different form of periodic waves and their interactions (superposition of waves), wave functions in quantum mechanics, etc.

In next class: Roots of Complex Numbers. Functions of Complex Variables. Analyticity and Cauchy-Riemann Conditions. Examples of analytic functions.

Bonus at the end: let us do some more problems appeared in previous years university examinations (from my age old note book):

Complex Variable Problems

A. Problems on simplification of complex numbers:

2014: Gr. A. 1. (a): Show that $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

2013: Gr. A. 1. (a). If the ratio be $\frac{z-i}{z-1}$ is purely imaginary, then show that the point z lies on a circle.

Gr. B: 2(a) Evaluate $\left| e^{-3iz+5i} \right|$

2012: 1(a) If n is positive integer, prove that $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6}$.

2011: 1(a) Find the angle θ for which $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is imaginary.

2009: 2(a) Simplify $\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

2008: 1(b) Find the smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$

2007: 2(b) Find the value of z for which $e^z = -5$

2006: 2(a) A rectangle on the complex z plane has its center at origin and sides parallel to the axes. If one vertex is at the point $\sqrt{2}+i$, find out the complex numbers representing the other vertices.

2005: Write down the equation of a circle of radius 1 with center at $(-1+i)$ in the complex plane.

2004: 1(a) Find the simplest form of $|e^{iz}|$, where $z = x+iy$.

2002: 2(d) Show that $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$

2001: 2(c) Express the following expression in the modulus amplitude form (i) $1+\sin x + i\cos x$.